SMOOTH MANIFOLDS FALL 2022 - HOMEWORK 5

Problem 1. Show that the distribution $E(p) = \{v \in \mathbb{R}^3 : \langle v, p \rangle = 0\}$ on $\mathbb{R}^3 \setminus \{0\}$ is involutive directly. What foliation does this integrate to? [*Hints*: Involutivity is a local property. You may use different vectors fields to frame at different points. Why won't you be able to find a single pair of vector fields to frame E globally? Finally, there are good and bad choices here. Part of the difficulty is finding "nice" fields. In particular, you can find linear fields.]

Problem 2. Prove the other direction of the Frobenius theorem. That is, show that if $\mathcal{F}$ is a foliation, then $E := T\mathcal{F}$ is involutive.

Problem 3. Let X and Y be vector fields on a smooth manifold M. Show that if $\varphi_t^X \circ \varphi_s^Y(x) = \varphi_{f(t,x)s}^Y \circ \varphi_t^X(x)$ for some smooth function $f : \mathbb{R} \times M \to \mathbb{R}$, all $s, t \in \mathbb{R}$ and $x \in M$, then $[X, Y]$ is a multiple of Y. Compute it in terms of the derivatives of f.

Non-graded

Problem 4. Let M and N be compact C^∞ manifolds of dimension m and n, respectively, and $\pi : M \to N$ be a submersion.

(1) Show that for every $y \in N$, $\pi^{-1}(y)$ is a compact manifold of dimension N. [This should be an immediate conclusion from a previous theorem!]

(2) Fix a chart $\psi : U \to \mathbb{R}^n$ for N such that $\psi(y) = 0$. Show that (by shrinking U as necessary), there exist n linearly independent vector fields $X_1, \dots, X_n$ defined on U and vector fields $V_1, \dots, V_n$ defined on $\pi^{-1}(U)$ such that
- $E = \operatorname{span}\{V_1, \dots, V_n\}$ is an n-dimensional distribution transverse such that $T_xM = T_x\pi^{-1}(y) \oplus E(x)$ for all $x \in \pi^{-1}(y)$,
- $d\pi(V_i(x)) = X_i(\pi(x))$ for all $x \in \pi^{-1}(U)$, and
- $d\psi(X_i(z)) = \partial/\partial x_i$ for all $z \in U$.

(3) Define the map $F : \pi^{-1}(y) \times B_{\mathbb{R}^n}(0, \varepsilon) \to M$ by

$$F(x, t_1, \dots, t_{m-n}) = \varphi_{t_{m-n}}^{V_{m-n}} \circ \dots \circ \varphi_{t_1}^{V_1}(x).$$

Show that $dF(x, 0)$ is invertible for every $x \in \pi^{-1}(y)$. Conclude that F is a diffeomorphism onto its image when restricted to a sufficiently small neighborhood of $\pi^{-1}(y) \times \{0\}$.

(4) Show that if $p = \psi \circ \pi \circ F$, then p is the projection onto $B_{\mathbb{R}^n}(0, \varepsilon)$-coordinate.